Covariance and correlation between the payout outcome of time-overlapping binary options with differing strike prices on the same underlying asset

Consider two binary option contracts on the same underlying asset S:

i. A cash or nothing call option with strike price K_1 and time to expiry $a + b$

ii. A cash or nothing call option with strike price K_2 and time to expiry $b + c$.

The options are simultaneously active for a time length of b, and option(ii) starts a time length of a ahead of option (i).

Let X and Y be the payout outcomes (including purchase prices) of both options respectively. One can model X and Y as random variables which take the value W_1 & W_2 with probability p_1 & p_2 and value L_1 & L_2 with probability $(1 - p_1)$ & $(1 - p_2)$ respectively.

<u>Assuming S follows a geometric brownian motion (GBM) process with drift rate r (equivalent to the risk-free rate) and volatility σ, what is the correlation between the payout outcomes?</u>

Using the gaussian increment property of a GBM process, one can model the variation of the log returns of S for a specific timeframe as a sum of normal iid random variables. Let (1). $Z_1\sigma\sqrt{a} + Z_2\sigma\sqrt{b}$ and (2). $Z_2\sigma\sqrt{b} + Z_3\sigma\sqrt{c}$ describe the log returns of S during the active timeframe of options (i) and (ii) respectively, where Z_1, Z_2, and Z_3 are standard normal random variables. $Z_2\sigma\sqrt{b}$ is the variation of the log returns when both options are active simultaneously, and thus present in both (1) and (2). The covariance and correlation between (1) and (2) are as follow:

$$\boldsymbol{Covariance}\left(\boldsymbol{Z_1\sigma\sqrt{a} + Z_2\sigma\sqrt{b}, Z_2\sigma\sqrt{b} + Z_3\sigma\sqrt{c}}\right)$$

$$= E\left[\left(Z_1\sigma\sqrt{a} + Z_2\sigma\sqrt{b}\right)\left(Z_2\sigma\sqrt{b} + Z_3\sigma\sqrt{c}\right)\right] - E\left(Z_1\sigma\sqrt{a} + Z_2\sigma\sqrt{b}\right)E\left(Z_2\sigma\sqrt{b} + Z_3\sigma\sqrt{c}\right)$$

$$= E(Z_1Z_2\sigma^2\sqrt{ab}) + E(Z_2^2\sigma^2 b) + E(Z_2Z_3\sigma^2\sqrt{bc}) + E(Z_1Z_3\sigma^2\sqrt{ac}) - 0$$

$$= \sigma^2 b \qquad (3).$$

$$\boldsymbol{Correlation}\left(\boldsymbol{Z_1\sigma\sqrt{a} + Z_2\sigma\sqrt{b}, Z_2\sigma\sqrt{b} + Z_3\sigma\sqrt{c}}\right)$$

$$= \frac{Covariance\left(Z_1\sigma\sqrt{a} + Z_2\sigma\sqrt{b}, Z_2\sigma\sqrt{b} + Z_3\sigma\sqrt{c}\right)}{\sqrt{Var\left(Z_1\sqrt{a} + Z_2\sqrt{b}\right)Var(Z_2\sqrt{b} + Z_3\sqrt{c})}}$$

$$= \frac{b}{\sqrt{(a+b)(b+c)}} \qquad (4).$$

With the knowledge of (3) which is a constant, one is able to model $\boldsymbol{B} = (B_1, B_2)^T$, the pair of log returns of S during the active timeframe of option (i) and (ii) perfectly with a bivariate normal

distribution with mean vector $\boldsymbol{\mu} = \left(\left(r - \frac{\sigma^2}{2}\right)(a+b), \left(r - \frac{\sigma^2}{2}\right)(b+c)\right)^T$, and covariance matrix $\Sigma = \begin{pmatrix} \sigma^2(a+b) & \sigma^2 b \\ \sigma^2 b & \sigma^2(b+c) \end{pmatrix}$. Thus, the covariance and correlation of X & Y are as follow:

$\boldsymbol{Covariance(X,Y)}$

$= E(XY) - E(X)E(Y)$

$= W_1W_2\ P(Both\ win) + W_1L_2P(first\ wins, second\ loses) + L_1W_2P(first\ loses, second\ wins) + L_1L_2P(Both\ lose) - (W_1p_1 + L_1(1-p_1))(W_2p_2 + L_2(1-p_2))$ (5).

To evaluate (5), one needs to calculate the probability weights for all XY outcomes. Let S_0 be the price of S when option (i) becomes active. Also, let K_1 be $S_0e^{-\Phi^{-1}(p_1)\sigma\sqrt{a+b} + \left(r-\frac{\sigma^2}{2}\right)(a+b)}$ and K_2 be $S_0e^{-\Phi^{-1}(p_2)\sigma\sqrt{b+c} + \left(r-\frac{\sigma^2}{2}\right)(b+c)}$ which are implied strike prices of the options given their probabilities of receiving the payout (p_1 and p_2). Then

$\boldsymbol{P(Both\ win)} = P(S_0e^{B_1} \geq S_0e^{-\Phi^{-1}(p_1)\sigma\sqrt{a+b} + \left(r-\frac{\sigma^2}{2}\right)(a+b)}\ \&\ S_0e^{B_2} \geq S_0e^{-\Phi^{-1}(p_2)\sigma\sqrt{b+c} + \left(r-\frac{\sigma^2}{2}\right)(b+c)})$

$= P(\frac{B_1 - \left(r-\frac{\sigma^2}{2}\right)(a+b)}{\sigma\sqrt{a+b}} \geq -\Phi^{-1}(p_1)\ \&\ \frac{B_2 - \left(r-\frac{\sigma^2}{2}\right)(b+c)}{\sigma\sqrt{b+c}} \geq -\Phi^{-1}(p_2))$

$= P(\frac{B_1 - \left(r-\frac{\sigma^2}{2}\right)(a+b)}{\sigma\sqrt{a+b}} \geq -\Phi^{-1}(p_1)\ \&\ \frac{B_2 - \left(r-\frac{\sigma^2}{2}\right)(b+c)}{\sigma\sqrt{b+c}} \geq -\Phi^{-1}(p_2))$

$= P(\frac{B_1 - \left(r-\frac{\sigma^2}{2}\right)(a+b)}{\sigma\sqrt{a+b}} < \Phi^{-1}(p_1)\ \&\ \frac{B_2 - \left(r-\frac{\sigma^2}{2}\right)(b+c)}{\sigma\sqrt{b+c}} < \Phi^{-1}(p_2))$

$= \int_{-\infty}^{\Phi^{-1}(p_2)} \int_{-\infty}^{\Phi^{-1}(p_1)} \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(B_1^2 - 2\rho B_1B_2 + B_2^2)\right) dB_1dB_2$

$= p_{ww}$

$\boldsymbol{P(first\ wins, second\ loses)} = P(first\ wins) - p_{ww} = p_1 - p_{ww}$

$\boldsymbol{P(first\ loses, second\ wins)} = P(second\ wins) - p_{ww} = p_2 - p_{ww}$

$\boldsymbol{P(both\ lose)} = 1 - p_2 - p_1 + p_{ww}$

where $\rho = \frac{b}{\sqrt{(a+b)(b+c)}}$.

Thus, resuming from (5):

$W_1W_2\ P(Both\ win) + W_1L_2P(first\ wins, second\ loses) + L_1W_2P(first\ loses, second\ wins) + L_1L_2P(Both\ lose) - (W_1p_1 + L_1(1-p_1))(W_2p_2 + L_2(1-p_2))$

$$= W_1W_2p_{ww} + W_1L_2(p_1 - p_{ww}) + L_1W_2(p_2 - p_{ww}) + L_1L_2(1 - p_2 - p_1 + p_{ww}) - (W_1p_1 + L_1(1 - p_1))(W_2p_2 + L_2(1 - p_2))$$

$$= W_1W_2p_{ww} + W_1L_2(p_1 - p_{ww}) + L_1W_2(p_2 - p_{ww}) + L_1L_2(1 - p_2 - p_1 + p_{ww}) - (W_1p_1 + L_1(1 - p_1))(W_2p_2 + L_2(1 - p_2))$$

$$= (W_1 - L_1)(W_2 - L_2)p_{ww} + W_1L_2p_1 + L_1W_2p_2 + L_1L_2(1 - p_2 - p_1) + W_1W_2p_1p_2 + W_2L_1p_2(1 - p_1) + W_1L_2p_1(1 - p_2) + L_1L_2(1 - p_1)(1 - p_2)$$

$$= (W_1 - L_1)(W_2 - L_2)p_{ww} + (W_1 - L_1)(W_2 - L_2)p_1p_2$$

$$= (W_1 - L_1)(W_2 - L_2)(p_{ww} - p_1p_2) \quad (6).$$

$\boldsymbol{Correlation(X, Y)}$

$$= \frac{Covariance(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

$$= \frac{(W_1 - L_1)(W_2 - L_2)(p_{ww} - p_1p_2)}{\sqrt{\left(E(X^2) - \left(E(X)\right)^2\right)\left(E(Y^2) - \left(E(Y)\right)^2\right)}}$$

$$= \frac{(W_1 - L_1)(W_2 - L_2)(p_{ww} - p_1p_2)}{\sqrt{(W_1^2p_1 + L_1(1 - p_1) - W_1^2p_1^2 - L_1^2(1 - p_1)^2 - 2W_1L_1p_1(1 - p_1))\left(E(Y^2) - \left(E(Y)\right)^2\right)}}$$

$$= \frac{(W_1 - L_1)(W_2 - L_2)(p_{ww} - p_1p_2)}{\sqrt{(W_1 - L_1)p_1(1 - p_1)\left(E(Y^2) - \left(E(Y)\right)^2\right)}}$$

$$= \frac{(W_1 - L_1)(W_2 - L_2)(p_{ww} - p_1p_2)}{\sqrt{(W_1 - L_1)(W_2 - L_2)p_1(1 - p_1)p_2(1 - p_2)}}$$

$$= \frac{(p_{ww} - p_1p_2)}{\sqrt{p_1(1 - p_1)p_2(1 - p_2)}} \quad (7).$$